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Non-Newtonian Fluid Flow on a Flat Plate Part 2: Heat Transfer

M. Mamun Molla*
University of Glasgow, Glasgow, Scotland G12 8QQ, United Kingdom
and
Lun-Shin Yao†
Arizona State University, Tempe, Arizona 85287

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Forced convective heat transfer of non-Newtonian fluids on a flat plate is investigated using a recently proposed modified power-law model. For a shear-thinning fluid, non-Newtonian effects are illustrated via local temperature distributions, heat transfer rate, and surface temperature distribution. Most significant effects occur near the leading edge, gradually tailing off far downstream.

Nomenclature

C	=	constant
D	=	nondimensional viscosity of the fluid
K	=	thermal conductivity
L	=	reference length of the plate
N	=	non-Newtonian power-law index
Q	=	timescale for the uniform surface heat-flux case
q_w	=	uniform surface heat flux
Re	=	Reynolds number
T	=	dimensional temperature of the fluid
T_w	=	surface temperature
T_∞	=	ambient temperature
U_0	=	freestream velocity
(U, V)	=	dimensionless fluid velocities in the (ξ, η) directions
$(\bar{u}, \bar{v})$	=	fluid velocities in the $(\bar{x}, \bar{y})$ directions
α	=	thermal diffusivity
γ	=	shear rate
η	=	pseudosimilarity variable
θ	=	dimensionless temperature of the fluid
ν	=	viscosity of the non-Newtonian fluid
ν_1	=	reference viscosity of the fluid
ξ	=	axial direction along the plate
ρ	=	fluid density

I. Introduction

THE interest in heat transfer problems involving power-law non-Newtonian fluids has grown persistently in the past half-century. It appears that Acrivos [1], a frequently cited paper, was the first to consider boundary-layer flows for such fluids. This interest in heat transfer problems is due to their wide relevance in chemicals, foods, polymers, molten plastics and petroleum production, and other natural phenomena [2–9].

Two common mistakes related to using a power-law correlation in previous boundary-layer formulations have been described recently in [10]. The first is that the explicit dependence of boundary layer development on streamwise coordinate has been arbitrarily ignored without justification. The correct approach makes note of the fact that because the similarity solution is valid at the leading edge of the flat plate, it should also be used as the upstream condition for the two-dimensional boundary-layer equations. The second concern is related to the unrealistic physical results, introduced by the power-law correlation, that viscosity either vanishes or becomes infinite at the limit of large or small shear rates, respectively. This condition introduces a nonremovable singularity at the leading edge or along the outer edge of boundary layers. Without recognizing the cause of such unrealistic conditions, complex multilayer structures have sometimes been introduced to overcome mathematical difficulties to obtain solutions of a nonphysical formulation [11,12].

By introducing a modified power-law correlation, these issues can be easily avoided [10]. Adopting this newly proposed correlation, we investigate the forced-convective heat transfer for a shear-thinning power-law non-Newtonian fluid with a large Prandtl number when a flat plate is heated at a constant surface temperature (CST) or with a uniform surface heat flux (USHF). The non-Newtonian effect is significant near the leading edge, gradually tailing off far downstream.

II. Formulation of the Problem

A steady laminar boundary layer of a non-Newtonian fluid along a semi-infinite heated flat plate was studied. The viscosity depends on shear rate and is correlated by a modified power law. We consider a shear-thinning situation and two different heating conditions. The first one is that the plate is heated at a CST T_w; the second uses a USHF q_w. The coordinate system is shown in Fig. 1.

Equations governing the flow and heat transfer are

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial}{\partial \bar{y}}\left(\nu \frac{\partial \bar{u}}{\partial \bar{y}}\right) \tag{2}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}} + \bar{v}\frac{\partial T}{\partial \bar{y}} = \alpha \frac{\partial^2 T}{\partial \bar{y}^2} \tag{3}$$

where $(\bar{u}, \bar{v})$ are velocity components along the $(\bar{x}, \bar{y})$ axes, T is the temperature, and α is the thermal diffusivity of the fluid. The viscosity is correlated by a modified power law, which is

$$\nu = \frac{K}{\rho}\left|\frac{\partial \bar{u}}{\partial \bar{y}}\right|^{n-1} \quad \text{for } \bar{\gamma}_1 \le \left|\frac{\partial \bar{u}}{\partial \bar{y}}\right| \le \bar{\gamma}_2 \tag{4}$$

The constants $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are two threshold shear rates, ρ is the density of the fluid, and K is a dimensional constant for which the dimension depends on the power-law index n. The values of these constants can be determined by matching with measurements. Outside of the preceding range, viscosity is assumed to be constant; its value can be fixed with data.

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*Department of Mechanical Engineering.
†Department of Mechanical and Aerospace Engineering.

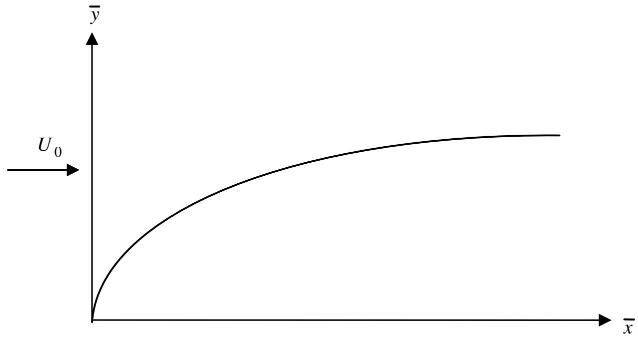

Fig. 1 Coordinates.

The boundary conditions for the present problem are as follows:

$$\bar{u} = \bar{v} = 0 \tag{5a}$$

For CST:

$$T = T_w \tag{5b}$$

For USHF:

$$-k\frac{\partial T}{\partial \bar{y}} = q_w \quad \text{at } \bar{y} = 0 \tag{5c}$$

where $\bar{u} \to U_0$ and $T \to T_\infty$ as $\bar{y} \to \infty$. The required upstream conditions will be described subsequently. We now introduce the following nondimensional variables and transform boundary-layer equations to parabolic coordinates (ξ, η):

$$\begin{aligned} \xi = x = \frac{\bar{x}}{l}, \qquad \eta = \bar{y}\left(\frac{U_0}{2\bar{x}\nu_1}\right)^{1/2}, \qquad U = \frac{\bar{u}}{U_0} \\ V = \bar{v}\left(\frac{2\bar{x}}{U_0\nu_1}\right)^{1/2}, \qquad Re = \frac{U_0 l}{\nu_1}, \qquad D = \frac{\nu}{\nu_1} \end{aligned} \tag{6a}$$

For CST:

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{6b}$$

For USHF:

$$\theta = (T - T_\infty)\frac{q_w}{k}\left(\frac{\nu_1 l}{U_0}\right)^{1/2} \tag{6c}$$

where ν_1 is the reference viscosity, θ is the dimensionless temperature of the fluid, and Re is the Reynolds number. The length scale associated with the non-Newtonian power law [10] is

$$l = U_0^{\frac{1}{2}}\left[\left(\frac{K}{\rho C}\right)\right]^{\frac{1}{n-1}} \upsilon_1^{\frac{n-3}{2(n-1)}} \tag{7}$$

Substituting the variables in Eq. (6) into Eqs. (1–4) leads to the following equations:

$$(2\xi)\frac{\partial U}{\partial \xi} - \eta\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \eta} = 0 \tag{8}$$

$$(2\xi)U\frac{\partial U}{\partial \xi} + (V - \eta U)\frac{\partial U}{\partial \eta} = \frac{\partial}{\partial \eta}\left[D\frac{\partial V}{\partial \eta}\right] \tag{9}$$

and

$$\gamma = (2\xi)^{-1/2}\frac{\partial U}{\partial \eta} \tag{10}$$

The dimensionless viscosity D is sketched in Fig. 2.

For CST,

$$\theta = \theta(\xi, \eta) \tag{11}$$

Using Eq. (11) in Eq. (3) yields

$$(2\xi)U\frac{\partial \theta}{\partial \xi} + (V - \eta U)\frac{\partial \theta}{\partial \eta} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} \tag{12}$$

For USHF,

$$\theta = (2\xi)^{-1/2}\theta(\xi, \eta) \tag{13}$$

Equation (3) takes the following form by using Eq. (13):

$$(2\xi)U\frac{\partial \theta}{\partial \xi} + (V - \eta U)\frac{\partial \theta}{\partial \eta} + U\theta = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} \tag{14}$$

Equations (8) and (9) can be solved by marching downstream, with the upstream condition satisfying the following differential equations:

$$-\eta\frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \eta} = 0 \tag{15}$$

$$(V - \eta U)\frac{\partial U}{\partial \eta} = \frac{\partial}{\partial \eta}\left[\frac{1}{2}\frac{\partial U}{\partial \eta}\right] \tag{16}$$

For CST:

$$(V - \eta U)\frac{\partial \theta}{\partial \eta} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} \tag{17a}$$

For USHF:

$$(V - \eta U)\frac{\partial \theta}{\partial \eta} + U\theta = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial \eta^2} \tag{17b}$$

which are the limiting Eqs. (8), (9), (12), and (14) as $\xi \to 0$. Then we can easily solve Eqs. (17) for the temperature function θ by marching downstream. The corresponding boundary conditions for Eqs. (15–17) are

$$U = V = 0 \tag{18a}$$

For CST:

$$\theta = 1 \tag{18b}$$

For USHF:

$$\frac{\partial \theta}{\partial \eta} = -1 \quad \text{at } \eta = 0 \tag{18c}$$

where $U \to 1$ and $\theta \to 0$ as $\eta \to \infty$. Equations are discretized by a central-difference scheme for the diffusion term and by a backward-difference scheme for the convection terms; finally, we get a system

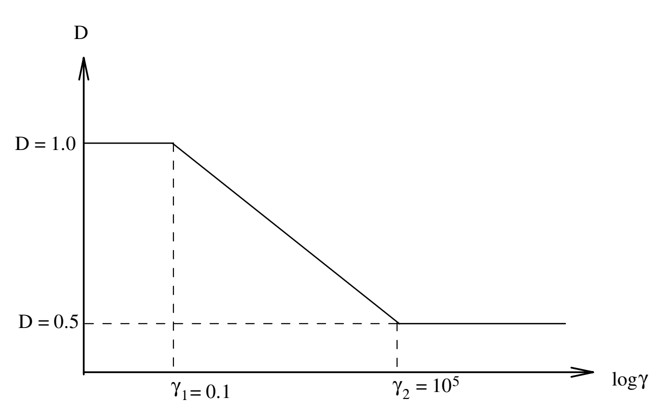

Fig. 2 Modified power-law correlation.

of implicit tridiagonal algebraic system of equations. The algebraic equations were solved by a double-sweep technique. In the computation, the continuity equation is directly solved for the normal velocity V. Hence, the truncation errors are $\mathcal{O}(\Delta\xi)$. The computation is started from $\xi = 0$ and then marches downstream to $\xi = 100$. After several test runs, convergent results are obtained by using $\Delta\xi = 2 \times 10^{-9}$ and $\Delta\eta = 0.001$ near the leading edge (say, $\xi = 0.0$–10^{-5}); afterward, $\Delta\xi$ is gradually increased to $\Delta\xi = 0.01$.

III. Results and Discussion

The numerical results are presented for the case of a shear thinning non-Newtonian fluid. In the following, the temperature distribution, the Nusselt number, and the surface temperature distribution are described.

For CST, the temperature distribution as a function of η at selected ξ locations is depicted in Figs. 3 and 4 for the Prandtl numbers 100 and 1000, respectively. The temperature distribution approaches that of the Newtonian fluid as the fluid moves downstream, where the shear stress decreases due to enhanced viscous effects (see [10]). The heat transfer rate in terms of the Nusselt number $Nu(2\xi)^{-1/2} = -\theta'(\xi, 0)$ is shown in Figs. 5 and 6 for $Pr = 100$ and 1000, respectively. The maximum heat transfer rate occurs at the leading edge for both Prandtl numbers. Their values are 2.49469 and 5.37653; the corresponding values for a Newtonian fluid are 2.22298 and 4.79007. Downstream from the leading edge, the heat transfer rate decreases rapidly up to $\xi \approx 12.0$, then slowly increases to approach the value of the Newtonian fluid.

For USHF, the temperature distributions are plotted in Figs. 7 and 8. The surface temperature distribution $\theta(\xi, 0)$ is provided in Figs. 9 and 10 for $Pr = 100$ and 1000, respectively. In these cases, the temperature distribution also approaches the temperature distribution of the Newtonian fluid downstream from the leading edge of the plate. The surface temperatures at $\xi = 0$ for the non-Newtonian fluid, which cannot be accurately read from the figures, are 0.29277 and 0.13587 for $Pr = 100$ and 1000, respectively, smaller than the corresponding values of 0.32859 and 0.15250 for a Newtonian fluid. This is due to the minimum viscosity; consequently, the maximum velocity occurs at the leading edge for the non-Newtonian fluid, as demonstrated in [10].

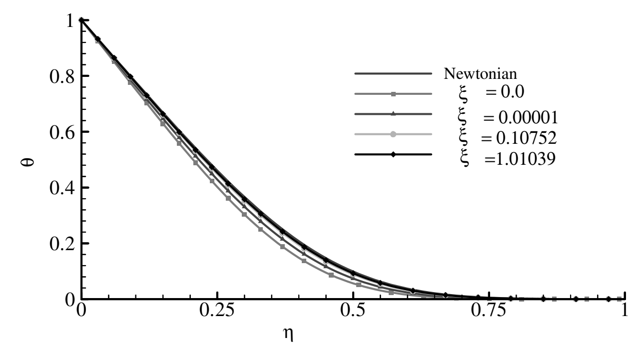

Fig. 3 Temperature distribution for *Pr* = 100.

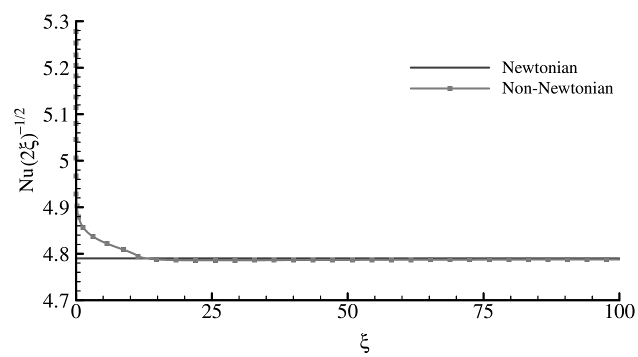

Fig. 6 Nusselt number for *Pr* = 1000.

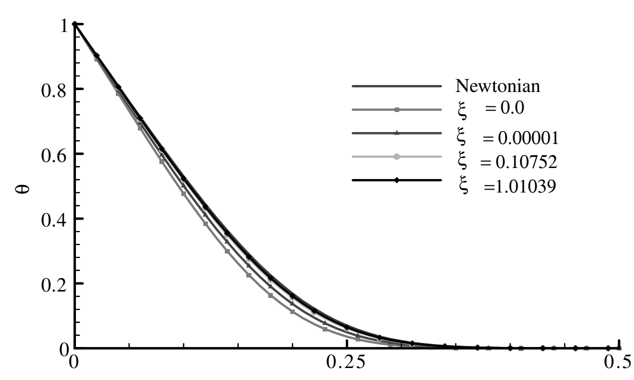

Fig. 4 Temperature distribution for *Pr* = 1000.

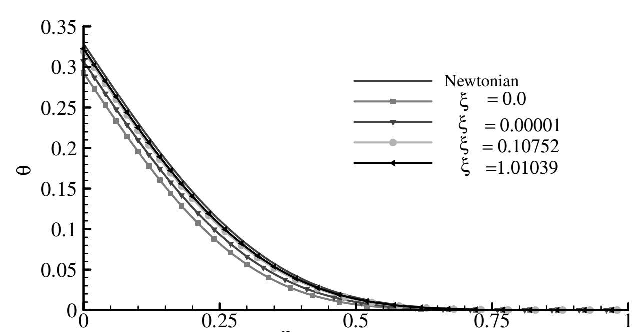

Fig. 7 Temperature distribution for *Pr* = 100.

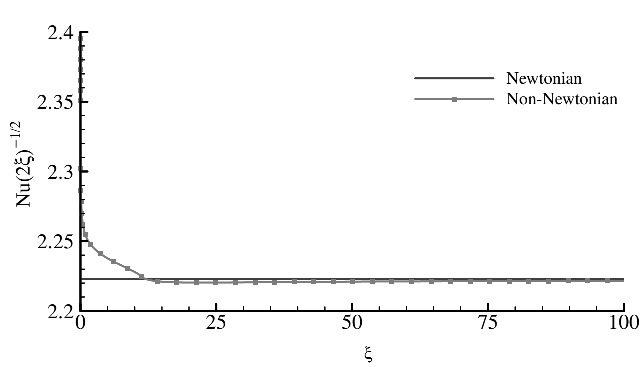

Fig. 5 Nusselt number for *Pr* = 100.

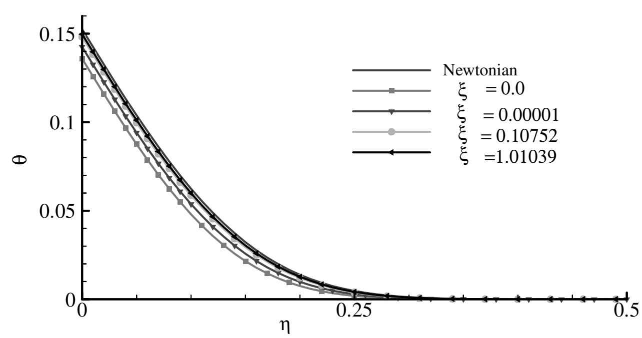

Fig. 8 Temperature distribution for *Pr* = 1000.

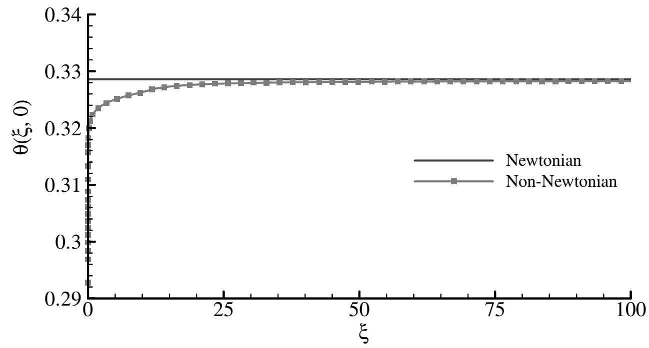

Fig. 9 Surface temperature distribution for $Pr = 100$.

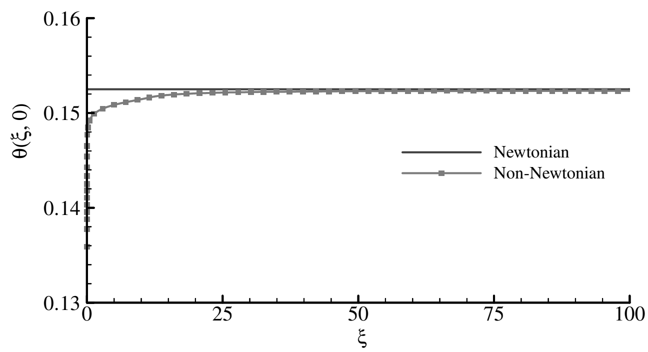

Fig. 10 Surface temperature distribution for $Pr = 1000$.

IV. Conclusions

We adopt the model for non-Newtonian fluids, recently proposed by [10], and apply it to the problem of forced convection on a flat plate. The modification for non-Newtonian fluids is based on actual measurements. The new model allows removal of the singularities at the leading edge of a flat-plate boundary layer for either shear-thinning or shear-thickening fluids. Under this condition, the boundary-layer and energy equations can be solved numerically by simple finite difference methods that march downstream from the leading edge, as is usually done for Newtonian fluids. The improvement of this new model for the prediction of fluid flows and energy transfer for problems other than boundary layers requires further study.

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